

# Vertex Operators of Admissible Modules of $U_q(C_n^{(1)})$

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## Abstract

Using our recent bosonic realization of  $U_q(\widehat{sp}_{2n})$ , we construct explicitly the vertex operators for the level  $-1/2$  modules of  $U_q(\widehat{sp}_{2n})$  using bosonic fields. Our method contains a detailed analysis of all the  $q$ -intertwining relations.

## 1 Introduction

The  $q$ -vertex operators play a crucial role in the mathematical formulation of the solvable lattice models (cf. [10]) and in the theory of  $q$ -KZ equations [8]. In particular the method works well when the explicit bosonization is available (cf. [16]).

The program of realization of vertex operators was carried out for many quantum affine algebras of classical types for both untwisted and twisted types at level one irreducible modules in [19] [14] [12] [15], and level two irreducible modules of  $U_q(\widehat{sl}_2)$  in [9] [14].

Recently we have given an explicit bosonic realization of the quantum affine algebra  $U_q(\widehat{sp}_{2n})$  by admissible representations of level  $-1/2$  in [13]. This realization is a  $q$ -analogue of the Feingold-Frenkel construction [4] of affine symplectic algebras and also generalizes the result of  $U_q(\widehat{sl}_2)$  in [20].

In this paper we use this explicit realization to construct  $q$ -vertex operators associated with the admissible representations. Admissible representations are generalizations of the integrable (integral levels) modules to those with levels of rational numbers [18]. Their characters are also modular functions.

Our construction has a new feature different from previous cases of positive levels. The intertwining relations are much harder to prove in our case due to appearance of poles in the contraction functions, while in the level one case they are consequences of some coproduct relations of Drinfeld generators. We give a detail analysis of the dependency among the intertwining relations and single out the key relations between the 0-vertex and the highest

component of the vertex operators. These key relations are then proved by a long chain of  $q$ -commutation relations of the algebra and the operators. The method clearly works for all other cases we mentioned above, and thus is completely general by its own means.

It is still unclear what physical models our vertex operators will describe. We only have an imbedding of  $V(\mu_i)$  into  $V(\mu_j) \otimes V_z$  instead of isomorphism in level one cases. It will be interesting to find out exactly how the level  $-1/2$ -modules are imbedded in  $\otimes^{\infty/2} V$ . It is clear that our  $q$ -vertex operator components generate a  $q$ -Weyl algebra. It would also be interesting to construct  $U_q(C_n^{(1)})$  from this  $q$ -Weyl algebra, which is regarded as one direction of the so-called boson-fermion correspondence (cf. [5]). In this aspect the work of [1][2] may be useful, where the similar problem for  $q$ -Clifford algebras was studied.

The paper is organized as follows. In section 2 we review the Feingold-Frenkel construction and discuss the equivalent form in terms of scalar fields. It is the latter form we quantized in [13] and recalled in section 3. The main results are also stated in section 3. In section 4 we start our analysis of the intertwining relations and reduce them to a few key relations, and point out that we only need to prove one of them. In the final section we prove the key relation.

## 2 Feingold-Frenkel construction of $\widehat{sp}_{2n}$ and vertex operators.

In this section, we review the Feingold-Frenkel construction of  $\widehat{sp}_{2n}$  in terms of the bosonic  $\beta$ - $\gamma$  system and describe the relations between the bosonic  $\beta$ - $\gamma$  system and vertex operators associated with the vector representation.

### 2.1 Affine Lie algebras $\widehat{sp}_{2n}$ .

We recall the affine Lie algebra  $\widehat{sp}_{2n}$  as follows. Most notations concerning affine Lie algebras follow [17]. Let  $V$  be a  $2n$ -dimensional  $\mathbf{C}$ -vector space with basis  $v_1, \dots, v_n, v_{\bar{1}}, \dots, v_{\bar{n}}$ . The Lie algebra  $sp_{2n}$  is defined as the Lie subalgebra of  $gl(V)$  generated by the elements:

$$\begin{aligned} E_i &= E_{ii+1} - E_{\overline{i+1}\bar{i}}, & F_i &= E_{i+1i} - E_{\overline{i}\bar{i+1}}, & E_n &= E_{n\bar{n}}, & F_n &= E_{\bar{n}n}, \\ H_i &= E_{ii} - E_{i+1i+1} + E_{\overline{i+1}\bar{i+1}} - E_{\bar{i}\bar{i}}, & H_n &= E_{nn} - E_{\bar{n}\bar{n}}. \end{aligned}$$

where  $E_{ij}, E_{\bar{i}\bar{j}}, \dots$  is the matrix unit of  $\text{End}(V)$  such that  $E_{ij}v_k = \delta_{jk}v_i$ , etc. We will refer to  $V$  as the vector representation of  $sp_{2n}$ . As a vector space  $\widehat{sp}_{2n}$  is given by

$$\widehat{sp}_{2n} = sp_{2n} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}K \oplus \mathbf{C}d.$$

The Lie algebra structure is defined by

$$\begin{aligned} [X \otimes t^l, Y \otimes t^m] &= [X, Y] \otimes t^{l+m} + l\delta_{l+m,0}Ktr(XY), & [X \otimes t^m, K] &= 0, \\ [d, K] &= 0, & [d, X \otimes t^m] &= mX \otimes t^m \quad \text{for } X, Y \in sp_{2n}. \end{aligned}$$

We set the following generating function for  $X \in sp_{2n}$ .

$$X(z) = \sum_{m \in \mathbf{Z}} (X \otimes t^m) z^{-m}.$$

## 2.2 Feingold-Frenkel construction.

As an index set, we set  $I = \mathbf{Z}$  or  $\mathbf{Z} + \frac{1}{2}$ . We introduce  $n$  copies of the bosonic  $\beta$ - $\gamma$  systems, which are the set of operators  $\beta_i(k)$ ,  $\gamma_i(k)$  ( $i = 1, \dots, n$ ,  $k \in I$ ) satisfying the defining relations:

$$[\beta_i(l), \gamma_j(m)] = \delta_{ij} \delta_{l+m, 0}, \quad [\beta_i(l), \beta_j(m)] = 0, \quad [\gamma_i(l), \gamma_j(m)] = 0,$$

for  $1 \leq i, j \leq n$ ,  $l, m \in I$ .

The Fock representation  $\mathcal{F}_{\beta-\gamma}$  is generated by the vacuum vector  $|vac\rangle$  with the following defining relations.

$$\beta_i(l)|vac\rangle = 0, \quad \gamma_i(m)|vac\rangle = 0 \quad \text{for } i = 1, \dots, n, \quad l \geq 0, \quad m > 0.$$

Let  $\beta_i(z)$ ,  $\gamma_i(z)$  be the generating functions:

$$\beta_i(z) = \sum_{k \in I} \beta_i(k) z^{-k}, \quad \gamma_i(z) = \sum_{k \in I} \gamma_i(k) z^{-k}.$$

Following [4], we define the action of  $\widehat{sp}_{2n}$  on  $\mathcal{F}_{\beta-\gamma}$  by

$$\begin{aligned} H_i(z) &= : \beta_i(z) \gamma_i(z) : - : \beta_{i+1}(z) \gamma_{i+1}(z) :, \\ E_i(z) &= \beta_i(z) \gamma_{i+1}(z), \quad F_i(z) = \beta_{i+1}(z) \gamma_i(z), \quad \text{for } i = 1, \dots, n-1, \\ H_n(z) &= - : \beta_n(z) \gamma_n(z) :, \quad E_n(z) = -\frac{1}{2} \beta_n(z) \beta_n(z), \quad F_n(z) = \frac{1}{2} \gamma_n(z) \gamma_n(z), \\ K &= -\frac{1}{2}, \quad d = -\sum_{i=1}^n \sum_{k \in I} k : \beta_i(k) \gamma_i(-k) :. \end{aligned}$$

where  $: \cdot :$  is the normal ordering defined by

$$: \beta_i(l) \gamma_i(m) : = \begin{cases} \beta_i(l) \gamma_i(m) & (l < m) \\ \frac{1}{2}(\beta_i(l) \gamma_i(m) + \gamma_i(m) \beta_i(l)) & (l = m) \\ \gamma_i(l) \beta_i(m) & (l > m) \end{cases}$$

Then we have the following theorem.

**Theorem 2.1** ([4])  $\mathcal{F}_{\beta-\gamma}$  is a  $\widehat{sp}_{2n}$ -module and decomposed into

$$L(-\frac{1}{2}\Lambda_0) \oplus L(-\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\delta) \quad \text{for } I = \mathbf{Z} + \frac{1}{2},$$

$$L(-\frac{1}{2}\Lambda_n) \oplus L(-\frac{3}{2}\Lambda_n + \Lambda_{n-1}) \quad \text{for } I = \mathbf{Z}.$$

Here  $L(\lambda)$  is the irreducible highest weight  $\widehat{sp}_{2n}$ -module with the highest weight  $\lambda$ ,  $\Lambda_i$  is the  $i$ -th fundamental weight and  $\delta$  is the canonical imaginary root. Highest weight vectors are given by  $|vac\rangle$ ,  $\beta_1(-\frac{1}{2})|vac\rangle$ ,  $|vac\rangle$ ,  $\gamma_n(0)|vac\rangle$  for  $L(-\frac{1}{2}\Lambda_0)$ ,  $L(-\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\delta)$ ,  $L(-\frac{1}{2}\Lambda_n)$ ,  $L(-\frac{3}{2}\Lambda_n + \Lambda_{n-1})$  respectively.

From now on we set  $\mu_1 = -\frac{1}{2}\Lambda_0$ ,  $\mu_2 = -\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\delta$ ,  $\mu_3 = -\frac{1}{2}\Lambda_n$ , and  $\mu_4 = -\frac{3}{2}\Lambda_n + \Lambda_{n-1}$ .

### 2.3 Vertex operators.

We first define  $V_z$ , the affinization of the vector representation  $V$ , which is a level 0 representation of  $\widehat{sp}_{2n}$ . As a vector space

$$V_z = V \otimes \mathbf{C}[z, z^{-1}].$$

The action of  $\widehat{sp}_{2n}$  is given by:

$$X(m).v \otimes z^n = X.v \otimes z^{m+n}, \quad d.v \otimes z^n = nv \otimes z^n, \quad K.v \otimes z^n = 0.$$

We define the operators  $\phi^V(z)$  as follows.

$$\phi^V(z) = \sum_{i=1}^n \gamma_i(z) \otimes v_i + \sum_{i=1}^n \beta_i(z) \otimes v_{\bar{i}},$$

which can be regarded as an operator from  $L(\mu)$  to a suitable completion of  $L(\mu') \otimes V_z$ , where  $(\mu, \mu') = (-\frac{1}{2}\Lambda_0, -\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\delta)$ ,  $(-\frac{3}{2}\Lambda_0 + \Lambda_1 - \frac{1}{2}\delta, -\frac{1}{2}\Lambda_0)$ ,  $(-\frac{1}{2}\Lambda_n, -\frac{3}{2}\Lambda_n + \Lambda_{n-1})$  or  $(-\frac{3}{2}\Lambda_n + \Lambda_{n-1}, -\frac{1}{2}\Lambda_n)$ . Then we have the following theorem.

**Theorem 2.2**  $\phi^V(z)$  is intertwining operator, or the vertex operators. (cf. [21], [8])

*Proof.* We can check the intertwining relations between  $\phi^V(z)$  and the  $\widehat{sp}_{2n}$  currents immediately.  $\square$

### 2.4 Representations in terms of scalar fields.

In this subsection, we explain the  $\beta$ - $\gamma$  system in terms of bosonic scalar fields ([7]). Using this picture, we can rewrite the Feingold-Frenkel construction in terms of the bosonic scalar fields. The rewritten representation coincides precisely with the classical limit ( $q = 1$ ) of the representation to be constructed in section 3. Let  $\phi_{i1}(z)$ ,  $\phi_{i2}(z)$  ( $i = 1, \dots, n$ ) be independent bosonic scalar fields normalized by

$$\phi_{i1}(z)\phi_{i1}(w) \sim -\log(z-w),$$

$$\phi_{i2}(z)\phi_{i2}(w) \sim \log(z-w).$$

It is known that the bosonic  $\beta$ - $\gamma$  system can be expressed by  $\phi_{i1}(z)$  and  $\phi_{i2}(z)$  as follows ([7]):

$$\begin{aligned}\beta_i(z) &\longrightarrow: \partial_z \phi_{i2}(z) e^{\phi_{i1}(z) + \phi_{i2}(z)}:, \\ \gamma_i(z) &\longrightarrow: e^{-\phi_{i1}(z) - \phi_{i2}(z)}:.\end{aligned}$$

We then obtain the Fock space of the bosonic  $\beta$ - $\gamma$  system by taking  $\text{Ker } Q_i^-$  for all  $i = 1, \dots, n$  with some charge constraints. Here  $Q_i^-$  is the operator  $\oint: e^{-\phi_{i2}(z)}: dz$ . The correspondence with  $Y_i(z)$  and  $Z_i(z)$  in section 3 is as follows:

$$Y_a^\pm(z) \xrightarrow{q \rightarrow 1}: e^{\pm 2(\phi_{i1}(z) - \phi_{i2}(z))}:, \quad Z_i^\pm(z) \longrightarrow: e^{\pm \phi_{i2}(z)}:.$$

### 3 Bosonic construction of $U_q(C_n^{(1)})$ and $q$ -vertex operators.

In this section, we review the bosonic construction of  $U_q(C_n^{(1)})$  in [13] and give a bosonization formula of the corresponding  $q$ -vertex operators.

#### 3.1 Quantum affine algebras $U_q(C_n^{(1)})$

Let  $\hat{P}$  be a free  $\mathbf{Z}$ -lattice of rank  $n + 2$  and we denote the basis by  $\varepsilon_1, \dots, \varepsilon_n, d, \delta$ . The nondegenerate inner product on  $\hat{P}$  is given by

$$(\varepsilon_i | \varepsilon_j) = \frac{1}{2} \delta_{ij}, \quad (d | \delta) = 1, \quad (d | d) = (\delta | \delta) = (\varepsilon_i | d) = (\varepsilon_i | \delta) = 0.$$

We set the simple roots  $\alpha_i$  and coroots  $h_i$  by

$$\alpha_0 = \delta - 2\varepsilon_1, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i = 1, \dots, n-1), \quad \alpha_n = 2\varepsilon_n,$$

$$h_0 = \delta - 2\varepsilon_1, \quad h_i = 2\varepsilon_i - 2\varepsilon_{i+1} \quad (i = 1, \dots, n-1), \quad h_n = 2\varepsilon_n.$$

Then the matrix  $((h_i | \alpha_j))_{0 \leq i, j \leq n}$  is the generalized Cartan matrix of type  $C_n^{(1)}$ . The fundamental weights  $\Lambda_0, \dots, \Lambda_n$  and their classical parts  $\lambda_1, \dots, \lambda_n$  are given by  $\Lambda_0 = d$ ,  $\Lambda_1 = d + \varepsilon_1$ ,  $\Lambda_2 = d + \varepsilon_1 + \varepsilon_2$ ,  $\dots$ ,  $\Lambda_n = d + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$  and  $\lambda_i = \Lambda_i - d$  ( $i = 1, \dots, n$ ). Let  $\hat{P}^*$ ,  $P$  and  $Q$  be the sublattices of  $\hat{P}$  defined as follows.

$$\begin{aligned}\hat{P}^* &= \mathbf{Z}h_0 \oplus \dots \oplus \mathbf{Z}h_n \oplus \mathbf{Z}d, \\ P &= \mathbf{Z}\varepsilon_1 \oplus \dots \oplus \mathbf{Z}\varepsilon_n = \mathbf{Z}\lambda_1 \oplus \dots \oplus \mathbf{Z}\lambda_n, \\ Q &= \mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_n.\end{aligned}\tag{1}$$

The quantum affine algebra  $U_q(C_n^{(1)})$  is the associative algebra with 1 over  $\mathbf{C}(q^{1/2})$  generated by the elements  $q^h$  ( $h \in \hat{P}^*$ ),  $e_i, f_i$ , ( $i = 0, 1, \dots, n$ ) satisfying the following defining relations.

$$q^h = 1 \quad \text{for } h = 0,$$

$$\begin{aligned}
q^{h+h'} &= q^h q^{h'} \quad \text{for } h, h' \in \hat{P}^*, \\
q^h e_i q^{-h} &= q^{(h|\alpha_i)} e_i \quad \text{and} \quad q^h f_i q^{-h} = q^{-(h|\alpha_i)} f_i, \\
[e_i, f_j] &= \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \\
\sum_{k=0}^b \frac{(-1)^k}{[k]_i! [b-k]_i!} e_i^k e_j e_i^{b-k} &= 0, \quad \sum_{k=0}^b \frac{(-1)^k}{[k]_i! [b-k]_i!} f_i^k f_j f_i^{b-k} = 0 \quad \text{for } i \neq j.
\end{aligned}$$

Here

$$b = 1 - (h_i|\alpha_j), \quad [k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad [k]_i! = [1]_i [2]_i \cdots [k]_i, \quad q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}} \quad \text{and} \quad t_i = q^{h_i} = q^{\alpha_i}.$$

In this paper, we use the following comultiplication.

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i.$$

Throughout this paper, we denote  $U_q(C_n^{(1)})$  by  $U_q$  and denote by  $V(\lambda)$  the irreducible highest weight  $U_q$ -module with highest weight  $\lambda$ . We fix a highest weight vector of  $V(\lambda)$  and denote it by  $|\lambda\rangle$ . If  $W_i$  ( $i = 1, 2$ ) has a weight decomposition  $W_i = \bigoplus_{\nu} W_{i,\nu}$ , their completed tensor product is then defined by

$$W_1 \hat{\otimes} W_2 = \bigoplus_{\nu} \left( \prod_{\nu=\nu_1+\nu_2} W_{1,\nu_1} \otimes W_{2,\nu_2} \right).$$

### 3.2 Bosonic construction of $U_q(C_n^{(1)})$ .

Let  $a_i(m)$  and  $b_i(m)$  be the operators satisfying the following defining relations.

$$\begin{aligned}
[a_i(m), a_j(l)] &= \delta_{m+l,0} \frac{[m(\alpha_i|\alpha_j)] [-\frac{m}{2}]}{m}, \\
[b_i(m), b_j(l)] &= m \delta_{ij} \delta_{m+l,0}, \\
[a_i(m), b_j(l)] &= 0.
\end{aligned} \tag{2}$$

We define the Fock space  $\mathcal{F}_{\alpha}^a$  and  $\mathcal{F}_{\beta}^{(b,i)}$  for  $\alpha \in P + \frac{\mathbb{Z}}{2}\lambda_n$ ,  $\beta \in P$  by the defining relations

$$a_i(m)|\alpha, \beta\rangle = 0 \quad (m > 0), \quad b_i(m)|\alpha, \beta\rangle = 0 \quad (m > 0),$$

$$a_i(0)|\alpha, \beta\rangle = (\alpha_i|\alpha)|\alpha, \beta\rangle, \quad b_i(0)|\alpha, \beta\rangle = (2\varepsilon_i|\beta)|\alpha, \beta\rangle,$$

where  $|\alpha, \beta\rangle$  is the vacuum vector. The grading operator  $d$  is defined by

$$d \cdot |\alpha, \beta\rangle = ((\alpha|\alpha) - (\beta|\beta - \lambda_n)) |\alpha, \beta\rangle.$$

We set

$$\tilde{\mathcal{F}} = \bigoplus_{\alpha \in P + \frac{1}{2}\mathbb{Z}\lambda_n, \beta \in P} \mathcal{F}_{\alpha, \beta}.$$

Let  $e^{a_i} = e^{\alpha_i}$  and  $e^{b_i}$  ( $1 \leq i \leq n$ ) be operators on  $\tilde{\mathcal{F}}$  given by:

$$e^{\varepsilon_i}|\alpha, \beta\rangle = |\alpha + \varepsilon_i, \beta\rangle \quad , \quad e^{b_i}|\alpha, \beta\rangle = |\alpha, \beta + \varepsilon_i\rangle.$$

Let  $:$  be the usual bosonic normal ordering defined by

$$: a_i(m)a_j(l) := a_i(m)a_j(l) \ (m \leq l), \ a_j(l)a_i(m) \ (m > l),$$

$$: e^a a_i(0) := a_i(0)e^a := e^a a_i(0).$$

and similar normal products for the  $b_i(m)$ 's. Let  $\partial = \partial_{q^{1/2}}$  be the  $q$ -difference operator:

$$\partial_{q^{1/2}}(f(z)) = \frac{f(q^{1/2}z) - f(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})z}$$

We introduce the following operators.

$$\begin{aligned} Y_i^\pm(z) &= \exp(\pm \sum_{k=1}^{\infty} \frac{a_i(k)}{[-\frac{1}{2}k]} q^{\pm \frac{k}{4}} z^k) \exp(\mp \sum_{k=1}^{\infty} \frac{a_i(k)}{[-\frac{1}{2}k]} q^{\pm \frac{k}{4}} z^{-k}) e^{\pm a_i} z^{\mp 2a_i(0)}, \\ Z_i^\pm(z) &= \exp(\pm \sum_{k=1}^{\infty} \frac{b_i(-k)}{k} z^k) \exp(\mp \sum_{k=1}^{\infty} \frac{b_i(k)}{k} z^{-k}) e^{\pm b_i} z^{\pm b_i(0)}. \end{aligned}$$

We define the operator  $x_i^\pm(m)$  ( $i = 1, \dots, n, m \in \mathbf{Z}$ ) by the following generating function  $X_i^\pm(z) = \sum_{m \in \mathbf{Z}} x_i^\pm(m) z^{-m-1}$ .

$$\begin{aligned} X_i^+(z) &= \partial Z_i^+(z) Z_{i+1}^-(z) Y_i^+(z), \quad i = 1, \dots, n-1 \\ X_i^-(z) &= Z_i^-(z) \partial Z_{i+1}^+(z) Y_i^-(z), \quad i = 1, \dots, n-1 \\ X_n^+(z) &= \left( \frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} : Z_n^+(z) \partial_z^2 Z_n^+(z) : - : \partial_z Y_b^+(q^{\frac{1}{2}}z) \partial_z Y_b^+(q^{-\frac{1}{2}}z) : \right) Y_a^+(z) \\ X_n^-(z) &= \frac{1}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} : Z_n^-(q^{\frac{1}{2}}z) Z_n^-(q^{-\frac{1}{2}}z) : Y_n^-(z). \end{aligned}$$

*Remark.* Our  $a_i(k)$  differs from that in [13], where we took  $a_i(k)/[d_i]$  for  $a_i(k)$ .

**Theorem 3.1** ([13])  $\tilde{\mathcal{F}}$  is a  $U_q$ -module of level  $-\frac{1}{2}$  under the action defined by

$$t_i \mapsto q^{\alpha_i}, \quad e_i \mapsto x_i^+(0), \quad f_i \mapsto x_i^-(0) \quad \text{for } i = 1, \dots, n,$$

$$t_0 \mapsto q^{-\frac{1}{2}} K_\theta^{-1} \quad (K_\theta = q^{2\alpha_1 + \dots + 2\alpha_{n-1} + \alpha_n}),$$

$$e_0 \mapsto \frac{(-1)^n}{[2]_1} [x_1^-(0), \dots, x_n^-(0), x_{n-1}^-(0), \dots, x_2^-(0), x_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1} q^{-1/2} \dots q^{-1/2} 1} K_\theta^{-1},$$

$$f_0 \mapsto \frac{(-q)^n}{[2]_1} [x_1^+(0), \dots, x_n^+(0), x_{n-1}^+(0), \dots, x_2^+(0), x_1^+(-1)]_{q^{-1/2} \dots q^{-1/2} q^{-1} q^{-1/2} \dots q^{-1/2} 1} K_\theta.$$

Furthermore, we know that  $\tilde{\mathcal{F}}$  contains the four irreducible highest weight modules ([13]). Let  $\mathcal{F}_{\alpha,\beta}^1$  be the subspace of  $\mathcal{F}_{\alpha,\beta}$  generated by  $a_i(m)$  ( $i = 1, \dots, n$  and  $m \in \mathbf{Z}$ ). Similarly, let  $\mathcal{F}_{\alpha,\beta}^{2,j}$  ( $j = 1, \dots, n$ ) be the subspace of  $\mathcal{F}_{\alpha,\beta}$  generated by  $b_j(m)$  ( $m \in \mathbf{Z}$ ). We can define the following isomorphism by  $|\alpha, \beta\rangle \otimes |\alpha', \beta'\rangle \rightarrow |\alpha + \alpha', \beta + \beta'\rangle$ .

$$\mathcal{F}_{\alpha,0}^1 \otimes \mathcal{F}_{0,\beta_1}^{2,1} \otimes \dots \otimes \mathcal{F}_{0,\beta_n}^{2,n} \longrightarrow \mathcal{F}_{\alpha,\beta_1+\dots+\beta_n}.$$

Let  $Q_j^-$  be the operator from  $\mathcal{F}_{\alpha,\beta}^{2,j}$  to  $\mathcal{F}_{\alpha,\beta-\varepsilon_j}^{2,j}$  defined by

$$Q_i^- = \frac{1}{2\pi\sqrt{-1}} \oint Z_i^-(z) dz.$$

We set subspaces  $\mathcal{F}_i$  ( $i = 1, 2, 3, 4$ ) of  $\tilde{\mathcal{F}}$  as follows.

$$\begin{aligned} \mathcal{F}_1 &= \bigoplus_{\alpha \in Q} \mathcal{F}'_{\alpha,\alpha}, & \mathcal{F}_2 &= \bigoplus_{\alpha \in Q} \mathcal{F}'_{\alpha+\varepsilon_1,\alpha+\varepsilon_1} \\ \mathcal{F}_3 &= \bigoplus_{\alpha \in Q} \mathcal{F}'_{\alpha-\frac{1}{2}\lambda_n,\alpha}, & \mathcal{F}_4 &= \bigoplus_{\alpha \in Q+\varepsilon_n} \mathcal{F}'_{\alpha-\frac{1}{2}\lambda_n,\alpha}, \end{aligned}$$

where

$$\mathcal{F}'_{\alpha,\beta} = \mathcal{F}_{\alpha,0}^1 \otimes \bigotimes_{j=1}^n \text{Ker}_{\mathcal{F}_{0,l_j\varepsilon_j}^{2,j}} Q_j^-,$$

for  $\beta = l_1\varepsilon_1 + \dots + l_n\varepsilon_n$ . Then we have the following theorem.

**Theorem 3.2** ([13]) *Each  $\mathcal{F}_i$  ( $i = 1, 2, 3, 4$ ) is an irreducible highest weight  $U_q$ -module isomorphic to  $V(\mu_i)$ , The highest weight vectors are given by  $|\mu_1\rangle = |0, 0\rangle$ ,  $|\mu_2\rangle = b_1(-1)|\lambda_1, \lambda_1\rangle$ ,  $|\mu_3\rangle = |-\frac{1}{2}\lambda_n, 0\rangle$ ,  $|\mu_4\rangle = |-\frac{1}{2}\lambda_n - \varepsilon_n, -\varepsilon_n\rangle$ .*

### 3.3 $q$ -vertex operators

We recall the evaluation  $U_q(C_n^{(1)})$ -module  $V_z$ . Let  $V$  be a  $2n$  dimensional vector space and  $v_1, \dots, v_n, v_{\overline{1}}, \dots, v_{\overline{n}}$  be basis of  $V$ . Set  $V_z = V \otimes \mathbf{C}[z, z^{-1}]$ . The action of  $U_q(C_n^{(1)})$  on  $V_z$  is defined as follows.

$$\begin{aligned} e_0 &= E_{\overline{1}\overline{1}} \otimes z, & e_i &= E_{ii+1} - E_{\overline{i+1}\overline{i}} \quad (\text{for } i = 1, \dots, n-1), & e_n &= E_{n\overline{n}}, \\ f_0 &= E_{1\overline{1}} \otimes z^{-1}, & f_i &= E_{i+1i} - E_{\overline{i}\overline{i+1}}, \quad (\text{for } i = 1, \dots, n-1), & f_n &= E_{\overline{n}n}, \\ q^h v \otimes z^n &= q^{(h|wt(v)+n\delta)} v \otimes z^n \quad \text{for } v = v_1, \dots, v_n, v_{\overline{1}}, \dots, v_{\overline{n}}, \end{aligned}$$

where  $wt(v)$  is given by  $wt(v_i) = \varepsilon_i$ ,  $wt(v_{\overline{i}}) = -\varepsilon_i$ .

**Definition 3.1** ([8]) *The  $q$ -vertex operator is a  $U_q$ -homomorphism of one of the following types.*

*Type I :*

$$\Phi_\lambda^{\mu V}(z) : V(\lambda) \longrightarrow V(\mu) \hat{\otimes} V_z$$

*Type II :*

$${}^V\Phi_\lambda^\mu(z) : V(\lambda) \longrightarrow V_z \hat{\otimes} V(\mu)$$



We define the components  $\Phi_\lambda^{\mu V}{}_i(z)$ ,  $\Phi_\lambda^{\mu V}{}_{\bar{i}}(z)$  of the  $q$ -vertex operators as follows.

$$\Phi_\lambda^{\mu V}(z) = \sum_{i=1}^n \Phi_\lambda^{\mu V}{}_i(z) \otimes v_i + \sum_{\bar{i}=1}^n \Phi_\lambda^{\mu V}{}_{\bar{i}}(z) \otimes v_{\bar{i}},$$

For the type II, the components are defined similarly.

We take the following normalization.

$$\Phi_{\mu_i}^{\mu_j V}(z)|\mu_i\rangle = |\mu_j\rangle \otimes v_{ij} + \text{terms of positive powers of } z, \quad (3)$$

where  $v_{12} = v_{\bar{1}}$ ,  $v_{21} = v_1$ ,  $v_{34} = v_n$ ,  $v_{43} = v_{\bar{n}}$ .

We also introduce the elements  $a_{\bar{1}}(k)$  in the Heisenberg algebra such that

$$[a_j(k), a_{\bar{1}}(l)] = \delta_{j1} \delta_{k,-l}$$

where

$$a_{\bar{1}}(k) = \sum_{i=1}^n \frac{k}{[k/2]^2} \frac{[(n+1-i)k][(n+1)k]_1}{[(n+1)k][(n+1-i)k]_i} a_i(k).$$

**Theorem 3.3** 1) For the type one vertex operators associated to  $(\lambda, \mu) = (\mu_1, \mu_2)$ ,  $(\mu_2, \mu_1)$ ,  $(\mu_3, \mu_4)$ , and  $(\mu_4, \mu_3)$  respectively, one has

$$\begin{aligned} \tilde{\Phi}_{\mu_i}^{\mu_j V}{}_{\bar{1}}(z) &= \exp\left(\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{(n+1/4-\delta_{n1})k} a_{\bar{1}}(-k) z^k\right) \exp\left(\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{-(n+3/4-\delta_{n1})k} a_{\bar{1}}(k) z^{-k}\right) \\ &\quad e^{\lambda_1 (q^{(n+1/2)} z)^{-\lambda_1(0)+1-(\lambda_1|\mu_i)}} \partial Z_1^+ (q^{n+1/2} z) c_{ij}, \\ \Phi_\lambda^{\mu V}{}_{\bar{j}+1}(z) &= -[\Phi_\lambda^{\mu V}{}_{\bar{j}}(z), f_j]_{q_j}, \quad \Phi_\lambda^{\mu V}{}_j(z) = [\Phi_\lambda^{\mu V}{}_{j+1}(z), f_j]_{q_j} \\ \Phi_\lambda^{\mu V}{}_n(z) &= [\Phi_\lambda^{\mu V}{}_{\bar{n}}(z), f_j]_{q_j}, \end{aligned} \quad (4)$$

where  $c_{ij}$  are constants for the four cases  $(\mu_i, \mu_j)$  with  $c_{12} = 1$ .

2) The type two vertex operators are given by

$$\begin{aligned} \Phi_\lambda^{\mu V}{}_1(z) &= \exp\left(-\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{-k/4} a_{\bar{1}}(-k) z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{[k/2]}{k} q^{-3k/4} a_{\bar{1}}(k) z^{-k}\right) c'_{ij} \\ &\quad e^{-\lambda_1 (q^{-1/2} z)^{\lambda_1(0)+1+(\lambda_1|\mu_i)}} Z_1^+ (q^{-1/2} z) c'_{ij}, \\ \Phi_\lambda^{\mu V}{}_{\bar{j}}(z) &= -[\Phi_\lambda^{\mu V}{}_{\bar{j}+1}(z), e_j]_{q_j}, \quad \Phi_\lambda^{\mu V}{}_{j+1}(z) = [\Phi_\lambda^{\mu V}{}_j(z), e_j]_{q_j}, \\ \Phi_\lambda^{\mu V}{}_{\bar{n}}(z) &= [\Phi_\lambda^{\mu V}{}_n(z), e_n]_q, \end{aligned}$$

where  $c'_{ij}$  are constants for the four cases  $(\mu_i, \mu_j)$  with  $c'_{12} = 1$ .

*Proof.* The proof of intertwining relations will be given in Section 4 and 5 on the space  $\tilde{\mathcal{F}}$ . We will only give the argument for the type I case, and type II case can be treated similarly word by word as in case I. It is obvious that each components of the vertex operators commute or anticommute with  $Q_j^-$ , thus the results are passed to the irreducible modules  $V(\mu_i)$ .  $\square$

## 4 Analysis of intertwining relations

The vertex operator  $\Phi(z) = \sum_{i=1}^n \Phi_i(z) \otimes v_i + \sum_{i=1}^n \Phi_{\bar{i}}(z) \otimes v_i$  is determined by the normalization (3) and the following relations: for  $i = 1, \dots, n$

$$t_j \Phi_i(z) t_j^{-1} = q_j^{\delta_{i,j+1} - \delta_{ij}} \Phi_i(z), \quad j = 0, 1, \dots, n \quad (5)$$

$$t_j \Phi_{\bar{i}}(z) t_j^{-1} = q_j^{\delta_{ij} - \delta_{i,j+1}} \Phi_{\bar{i}}(z), \quad j = 0, 1, \dots, n \quad (6)$$

$$[\Phi_i(z), e_j] = t_j \Phi_{j+1}(z) \delta_{ij}, \quad j = 1, \dots, n-1 \quad (7)$$

$$[\Phi_{\bar{i}}(z), e_j] = -t_j \Phi_{\bar{j}}(z) \delta_{i,j+1}, \quad j = 1, \dots, n \quad (8)$$

$$[\Phi_{\bar{i}}(z), e_0] = z t_0 \Phi_1(z) \delta_{i1}, \quad [\Phi_i(z), e_0] = 0, \quad j = 1, \dots, n, \quad (9)$$

$$[\Phi_i(z), e_n] = t_n \Phi_{\bar{n}}(z) \delta_{in}, \quad [\Phi_{\bar{i}}(z), e_n] = 0, \quad j = 1, \dots, n, \quad (10)$$

$$[\Phi_i(z), f_j]_{q_j^{\delta_{i,j+1} - \delta_{ij}}} = \Phi_j(z) \delta_{i,j+1}, \quad i, j = 1, \dots, n \quad (11)$$

$$[\Phi_{\bar{i}}(z), f_j]_{q_j^{\delta_{ij} - \delta_{i,j+1}}} = -\Phi_{\bar{j+1}}(z) \delta_{ij}, \quad i, j = 1, \dots, n \quad (12)$$

$$[\Phi_i(z), f_0]_{q^{\delta_{i1}}} = z^{-1} \Phi_{\bar{1}}(z) \delta_{i1}, \quad [\Phi_{\bar{i}}(z), f_0]_{q^{-\delta_{i1}}} = 0, \quad (13)$$

$$[\Phi_{\bar{i}}(z), f_n]_{q^{\delta_{in}}} = -\Phi_n(z) \delta_{in}, \quad [\Phi_{\bar{i}}(z), f_n]_{q^{\delta_{in}}} = 0 \quad (14)$$

With the construction of  $\Phi_{\bar{1}}(z)$  given in (4), we define the other components:

$$\Phi_{\bar{i+1}}(z) = -[\Phi_{\bar{i}}(z), f_i]_{q^{1/2}}, \quad i = 1, \dots, n-1 \quad (15)$$

$$\Phi_n(z) = [\Phi_{\bar{n}}(z), f_n]_q, \quad (16)$$

$$\Phi_i(z) = [\Phi_{i+1}(z), f_i]_{q^{1/2}}, \quad i = 1, \dots, n-1 \quad (17)$$

In order to study  $q$ -commutators, we recall the following identities from [11]:

$$[a, [b, c]_u]_v = [[a, b]_x, c]_{uv/x} + x [b, [a, c]_{v/x}]_{v/x}, \quad x \neq 0 \quad (18)$$

$$[[a, b]_u, c]_v = [a, [b, c]_x]_{uv/x} + x [[a, c]_{v/x}, b]_{u/x}, \quad x \neq 0 \quad (19)$$

**Proposition 4.1** *Let  $\Phi_i(z)$  be vertex operators defined by  $\Phi_{\bar{1}}(z)$  via (15, 16, 17) and satisfy the relations:*

$$[\Phi_{\bar{1}}(z), f_j]_{q_j^{\delta_{1j}}} = -\Phi_{\bar{2}}(z) \delta_{1j}, \quad [\Phi_{\bar{1}}(z), e_j] = 0, \quad j = 1, \dots, n$$

$$[f_1, [f_1, \Phi_{\bar{1}}(z)]_{q^{-1/2}}]_{q^{1/2}} = 0.$$

Then we have for  $i = 0, 1, \dots, n, j = 1, \dots, n$

$$[\Phi_i(z), e_j] = t_j \Phi_{j+1}(z) \delta_{ij}, \quad (20)$$

$$[\Phi_{\bar{i}}(z), e_j] = -t_j \Phi_{\bar{j}}(z) \delta_{i,j+1}, \quad (21)$$

$$[\Phi_i(z), e_n] = t_n \Phi_{\bar{n}}(z) \delta_{in}, \quad [\Phi_{\bar{i}}(z), e_n] = 0, \quad (22)$$

$$[\Phi_i(z), f_j]_{q_j^{\delta_{i,j+1} - \delta_{ij}}} = \Phi_j(z) \delta_{i,j+1}, \quad (23)$$

$$[\Phi_{\bar{i}}(z), f_j]_{q_j^{\delta_{ij} - \delta_{i,j+1}}} = -\Phi_{\bar{j+1}}(z) \delta_{ij}, \quad (24)$$

where we identify  $\Phi_{n+1}(z) = \Phi_{\bar{n}}(z)$  in (20),  $\Phi_{\bar{0}}(z) = -z\Phi_1(z)$  in (21),  $\Phi_{\overline{n+1}}(z) = \Phi_n(z)$  in (23).

*Proof.* Using relations of the quantum affine algebras  $U_q(C_n^{(1)})$ , we have for  $j \neq i-1$ :

$$\begin{aligned} [\Phi_{\bar{i}}(z), e_j] &= - \left[ [\Phi_{\overline{i-1}}(z), f_{i-1}]_{q^{1/2}}, e_j \right] \\ &= - \left[ [\Phi_{\overline{i-1}}(z), e_j], f_{i-1} \right]_{q^{1/2}} = 0 \quad \text{by induction} \end{aligned}$$

$$\begin{aligned} [\Phi_{\bar{i}}(z), e_{i-1}] &= - \left[ [\Phi_{\overline{i-1}}(z), f_{i-1}]_{q^{1/2}}, e_{i-1} \right] \\ &= \left[ \Phi_{\overline{i-1}}(z), \frac{t_{i-1} - t_{i-1}^{-1}}{q^{1/2} - q^{-1/2}} \right]_{q^{1/2}} = -t_{i-1}\Phi_{i-1}(z). \end{aligned}$$

$$\begin{aligned} [\Phi_i(z), e_j] &= \left[ [\Phi_{i+1}(z), f_i]_{q^{1/2}}, e_j \right] = - \left[ \Phi_{\overline{i+1}}(z), \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \right]_{q_i} \delta_{ij} \\ &= t_j \Phi_{j+1}(z) \delta_{ij}, \end{aligned}$$

where we used induction to get  $[\Phi_{i+1}, e_j] = 0$  for  $j \neq i$  in the second equality. Similarly we can prove the commutation relations of  $\Phi_i(z)$  and  $e_n$  for  $i = n, \bar{n}$ .

Now we consider the relations between  $\Phi_i(z)$  and  $f_j$ . It follows from (19) that

$$\begin{aligned} [\Phi_i(z), f_j] &= \left[ [\Phi_{i-1}(z), f_{i-1}]_{q^{1/2}}, f_j \right] \\ &= \left[ [\Phi_{i-1}(z), f_j], f_{i-1} \right]_{q^{1/2}} + [\Phi_{i-1}(z), [f_{i-1}, f_j]]_{q^{1/2}} \end{aligned}$$

Then the relation (11) is true: when  $i = j$  is the definition;  $i = j+1$  follows from Proposition 4.3, and other cases are derived from the relation.

$$\begin{aligned} [\Phi_{\bar{i}}(z), f_{i-1}]_{q^{-1/2}} &= - \left[ [\Phi_{\overline{i-1}}(z), f_{i-1}]_{q^{1/2}}, f_{i-1} \right]_{q^{-1/2}} \\ &= - \left[ \left[ [\Phi_{\overline{i-2}}(z), f_{i-2}]_{q^{1/2}}, f_{i-1} \right]_{q^{1/2}}, f_{i-1} \right]_{q^{-1/2}} \end{aligned}$$

Now let use induction on  $i$ . The initial step  $i = 2$  is Proposition (4.3). From the induction hypothesis  $[\Phi_{\overline{i-2}}(z), f_{i-1}]_{q^{1/2}} = 0$ , we see the above commutation turns to:

$$\begin{aligned} [\Phi_{\bar{i}}(z), f_{i-1}]_{q^{-1/2}} &= - [\Phi_{\overline{i-2}}(z), [f_{i-2}, f_{i-1}]_{q^{1/2}}, f_{i-1}]_{q^{-1/2}} \\ &= 0. \quad \text{by Serre relation} \end{aligned}$$

Clearly we have

$$[\Phi_{\bar{i}}(z), f_{i+1}] = - \left[ [\Phi_{\overline{i-1}}(z), f_{i-1}]_{q^{1/2}}, f_{i+1} \right] = 0.$$

$$\begin{aligned}
[\Phi_{\overline{i}}(z), e_0] &= -[\Phi_{\overline{i-1}}(z), e_0], f_{i-1}] \\
&= -\delta_{i2}[t_0\Phi_1(z), f_1]_{q^{1/2}} = -\delta_{i2}t_0[\Phi_1, f_1]_{q^{-1/2}} \\
&= 0 \quad \text{by (11)}
\end{aligned}$$

where we used induction on  $i$  in the second equality. Similarly we can check that  $[\Phi_j(z), f_0] = 0$  for  $j \neq 1$  by induction.

Next we claim that  $[\Phi_{\overline{1}}(z), e_0] = zt_0\Phi_1(z)$  is equivalent to  $[\Phi_1(z), f_0]_{q^{-1}} = \Phi_{\overline{1}}(z)z^{-1}$ . In fact, given the former we have that

$$\begin{aligned}
[\Phi_1(z), f_0] &= z^{-1} [t_0^{-1}[\Phi_{\overline{1}}(z), e_0], f_0] \\
&= z^{-1}t_0^{-1} [[\Phi_{\overline{1}}(z), e_0], f_0]_{q^{-1}} \\
&= z^{-1}t_0^{-1} \left[ \Phi_{\overline{1}}(z), \frac{t_0 - t_0^{-1}}{q - q^{-1}} \right]_{q^{-1}} = z^{-1}\Phi_{\overline{1}}(z).
\end{aligned}$$

With the relations of  $[\Phi_{\overline{i}}(z), f_j]_{q_j^{\delta_{ij} - \delta_{i,j+1}}} = -\Phi_{\overline{j+1}}(z)\delta_{ij}$  in hand, we can compute that

$$\begin{aligned}
[\Phi_i(z), f_i]_{q^{-1/2}} &= \left[ [\Phi_{i+2}(z), f_{i+1}]_{q_{i+1}}, f_i \right]_{q^{1/2}, f_i} \Big|_{q^{-1/2}} \\
&= \left[ \Phi_{i+2}(z), [f_{i+1}, f_i]_{q^{1/2}, f_i} \right]_{q^{-1/2}} \Big|_{q_{1/2}} \\
&= 0
\end{aligned}$$

where we have used induction on  $n - i$  and  $i \leq n - 2$ . When  $i = n, n - 1$  we can compute similarly. For example,

$$\begin{aligned}
[\Phi_n(z), f_n]_{q^{-1}} &= \left[ [\Phi_{\overline{n-1}}(z), f_{n-1}]_{q^{1/2}}, f_n \right]_{q, f_n} \Big|_{q^{-1}} \\
&= \left[ \Phi_{\overline{n-1}}(z), [f_{n-1}, f_n]_{q, f_n} \right]_{q^{-1}} \Big|_{q_{1/2}} \\
&= 0
\end{aligned}$$

The other cases of  $[\Phi_i(z), f_j]_{q_j^{-\delta_{ij} + \delta_{i,j+1}}} = \Phi_j(z)\delta_{i-1,j}$  are shown similarly.

**Proposition 4.2** *The operator  $\Phi(z)$  (cf. (4)) satisfy the following relations:*

$$\begin{aligned}
[\Phi_{\overline{1}}(z), X_i^+(w)] &= 0, \quad i = 1, \dots, n \\
[\Phi_{\overline{1}}(z), X_i^-(w)] &= 0, \quad i = 2, \dots, n \\
[\Phi_{\overline{1}}(z), X_1^-(w)]_{q^{-1/2}} &= \frac{z}{w} q^n [\Phi_{\overline{1}}(z), X_1^-(w)]_{q^{1/2}}
\end{aligned}$$

*Proof.* The first set of relations follow from our construction. We only need to see  $i = 1$  in the second set. Writing  $\Phi_{\overline{1}}(z) = \frac{1}{(q^{1/2} - q^{-1/2})z} (\Phi_{\overline{1+}}(z) - \Phi_{\overline{1-}}(z))$ , we have

$$\Phi_{\overline{1\epsilon}}(z)X_{1\epsilon'}^+(z) = : \Phi_{\overline{1\epsilon}}(z)X_{1\epsilon'}^+(z) : q^{\epsilon/2} \frac{q^{n+1/2}z - q^{\epsilon'/2 - \epsilon/2}w}{q^{n+1/2}z - w} \quad (25)$$

$$X_{1\epsilon'}^-(z)\Phi_{\overline{1\epsilon}}(z) = : \Phi_{\overline{1\epsilon}}(z)X_{1\epsilon'}^-(z) : q^{\epsilon/2} \frac{q^{\epsilon'/2 - \epsilon/2}w - q^{n+1/2}z}{q^{n+1/2}w - z}, \quad (26)$$

$$\begin{aligned}
& [\Phi_{\bar{1}}(z), X_i^+(w)] \\
&= \sum_{\epsilon=\pm 1} [\Phi_{\bar{1}\epsilon}(z), X_{i,-\epsilon}^+(w)] \\
&= \sum_{\epsilon=\pm 1} : \Phi_{\bar{1}\epsilon}(z) X_{1\epsilon'}^-(z) : \frac{q^{\epsilon/2}(q^{n+1/2}z - q^{-\epsilon}w)}{w} \delta\left(\frac{q^{n+1/2}z}{w}\right) \\
&= 0
\end{aligned}$$

since  $: \Phi_{\bar{1}+}(z) X_{1-}^+(w) := : \Phi_{\bar{1}-}(z) X_{1-}^+(w) :$ .

The third relation is another form of the following identity.

$$\begin{aligned}
& q^{1/2}(w - q^n z) \Phi_{\bar{1}}(z) X_1^-(w) + (q^{n+1}z - w) X_1^-(w) \Phi_{\bar{1}}(z) \\
&= \sum_{\epsilon', \epsilon=\pm} q^{1/2}(w - q^n z) \Phi_{\bar{1}\epsilon}(z) X_{1\epsilon'}^-(w) + (q^{n+1}z - w) X_{1\epsilon'}^-(w) \Phi_{\bar{1}\epsilon}(z) \\
&= \sum_{\epsilon', \epsilon=\pm} : \Phi_{\bar{1}\epsilon}(z) X_{1\epsilon'}^-(w) : q^{-\epsilon/2}(q^{n+1}z - w)(w - q^n z) \delta\left(\frac{q^{(2n+\epsilon+1)/2}z}{w}\right) q^{-n-\epsilon/2} z^{-1} \\
&= 0
\end{aligned}$$

□

In particular, we have

$$[\Phi_{\bar{1}}(z), X_1^-(1)]_{q^{-1/2}} = z q^n [\Phi_{\bar{1}}(z), X_1^-(0)]_{q^{1/2}} \quad (27)$$

**Proposition 4.3** *The operator constructed in (4) satisfies the following Serre-like relation:*

$$\Phi_{\bar{1}}(z) f_1^2 - (q^{1/2} + q^{-1/2}) f_1 \Phi_{\bar{1}}(z) f_1 + f_1^2 \Phi_{\bar{1}}(z) = 0$$

*Proof.* It follows from (25, 26) that

$$\begin{aligned}
\Phi_{\bar{1}\epsilon}(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) &= : \Phi_{\bar{1}\epsilon}(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) : \\
&\quad \prod_{i=1,2} \frac{q^{n+1/2}w - q^{-1/2}z_i}{q^{(2n+\epsilon+1)/2}w - z_i} \cdot \frac{q^{\epsilon_1/2}z_1 - q^{\epsilon_2/2}z_2}{z_1 - q^{-1}z_2} \\
X_{1\epsilon_1}^-(z_1) \Phi_{\bar{1}\epsilon}(w) X_{1\epsilon_2}^-(z_2) &= : \Phi_{\bar{1}\epsilon}(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) : \\
&\quad \frac{z_1 - q^n w}{z_1 - q^{(2n+\epsilon+1)/2}w} \frac{q^{n+1/2}w - q^{-1/2}z_2}{q^{(2n+\epsilon+1)/2}w - z_2} \cdot \frac{q^{\epsilon_1/2}z_1 - q^{\epsilon_2/2}z_2}{z_1 - q^{-1}z_2} \\
X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) \Phi_{\bar{1}\epsilon}(w) &= : \Phi_{\bar{1}\epsilon}(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) : \\
&\quad \prod_{i=1,2} \frac{z_i - q^n w}{z_i - q^{(2n+\epsilon+1)/2}w} \cdot \frac{q^{\epsilon_1/2}z_1 - q^{\epsilon_2/2}z_2}{z_1 - q^{-1}z_2}
\end{aligned}$$

Then we have

$$\Phi_{\bar{1}\epsilon}(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) - (q^{1/2} + q^{-1/2}) X_{1\epsilon_1}^-(z_1) \Phi_{\bar{1}}(w) X_{1\epsilon_2}(z_2) + X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) \Phi_{\bar{1}}(z)$$

$$\begin{aligned}
&= : \Phi_{\bar{1}\epsilon}^-(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_2}^-(z_2) : \frac{q^{\epsilon_1/2} z_1 - q^{\epsilon_2/2} z_2}{(z_1 - q^{-1} z_2) \prod_{i=1,2} (q^{(2n+\epsilon+1)/2} w - z_i)} \\
&\quad \cdot \left\{ (q^{n+1/2} w - q^{-1/2} z_1)(q^{n+1/2} w - q^{-1/2} z_2) - (q^{1/2} + q^{-1/2}) \cdot \right. \\
&\quad \left. \cdot (z_1 - q^n w)(q^{-1/2} z_2 - q^{n+1/2} w) + (z_1 - q^n w)(z_2 - q^n w) \right\} \\
&= : : \frac{q^{\epsilon_1/2} z_1 - q^{\epsilon_2/2} z_2}{\prod_{i=1,2} (q^{(2n+\epsilon+1)/2} w - z_i)} q^n (1 - q) w
\end{aligned}$$

Note that the contraction function is antisymmetric respect to  $z_1 \mapsto z_2$  when  $\epsilon_1 = \epsilon_2$ . Therefore,

$$\begin{aligned}
&Sym_{z_1, z_2} \Phi_{\bar{1}\epsilon}^-(w) X_{1\epsilon_1}^-(z_1) X_{1\epsilon_1}^-(z_2) \\
&\quad - (q^{1/2} + q^{-1/2}) X_{1\epsilon_1}^-(z_1) \Phi_{\bar{1}}^-(w) X_{1\epsilon_1}^-(z_2) + X_{1\epsilon_1}^-(z_1) X_{1\epsilon_1}^-(z_2) \Phi_{\bar{1}}^-(z) = 0
\end{aligned}$$

Furthermore when  $\epsilon_1 = -\epsilon_2$  we have

$$\begin{aligned}
&\sum_{\epsilon_1} \Phi_{\bar{1}\epsilon}^-(w) X_{1\epsilon_1}^-(z_1) X_{1-\epsilon_1}^-(z_2) - (q^{1/2} + q^{-1/2}) X_{1\epsilon_1}^-(z_1) \Phi_{\bar{1}}^-(w) X_{1-\epsilon_1}^-(z_2) + \\
&\quad X_{1\epsilon_1}^-(z_1) X_{1-\epsilon_1}^-(z_2) \Phi_{\bar{1}}^-(z) \\
&= : \Phi_{\bar{1}\epsilon}^-(w) X_{1\epsilon_1}^-(z_1) X_{1-\epsilon_1}^-(z_2) : \frac{(q^{1/2} + q^{-1/2})(z_1 - z_2)}{\prod_{i=1,2} (q^{(2n+\epsilon+1)/2} w - z_i)} q^n (1 - q) w
\end{aligned}$$

which is also antisymmetric with respect to  $z_1, z_2$ . We thus have that

$$\begin{aligned}
&Sym_{z_1, z_2} \Phi_{\bar{1}}^-(w) X_1^-(z_1) X_1^-(z_2) \\
&\quad - (q^{1/2} + q^{-1/2}) X_1^-(z_1) \Phi_{\bar{1}}^-(w) X_1^-(z_2) + X_1^-(z_1) X_1^-(z_2) \Phi_{\bar{1}}^-(z) = 0
\end{aligned}$$

Picking up coefficients of  $(z_1 z_2)^m$  we proved the identity.  $\square$

## 5 Proof of $[\Phi_{\bar{1}}(z), e_0] = z t_0 \Phi_1(z)$

In the last section we have shown that we are left to check only the relation in the section title. The key of the the proof are the  $q$ -commutation relations (18) and (19) (cf. [11]).

As in [11] we introduce the twisted commutators  $[b_1, \dots, b_n]_{v_1 \dots v_{n-1}}$  and  $[b_1, \dots, b_n]'_{v_1 \dots v_{n-1}}$  inductively by  $[b_1, b_2]_v = [b_1, b_2]'_v = b_1 b_2 - v b_2 b_1$  and

$$\begin{aligned}
[b_1, \dots, b_n]_{v_1 \dots v_{n-1}} &= [b_1, [b_2, \dots, b_n]_{v_1 \dots v_{n-2}}]_{v_{n-1}} \\
[b_1, \dots, b_n]'_{v_1 \dots v_{n-1}} &= [[b_1, \dots, b_{n-1}]'_{v_1 \dots v_{n-2}}, b_n]_{v_{n-1}}
\end{aligned}$$

In this notation we have that

$$\begin{aligned}
&e_0 \\
&= [X_1^-(0), \dots, X_n^-(0), X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1} q^{-1/2} \dots q^{-1/2} 1} \gamma K_{\theta}^{-1} \frac{(-1)^n}{[2]_1}.
\end{aligned}$$

Write  $\hat{e}_0 = [2]_1 e_0 \gamma^{-1} K_\theta$ , then the relation we want is

$$[\Phi_1^-(z), \hat{e}_0]_{q^{-1}} = (-1)^n (1+q) z \Phi_1(z).$$

**Lemma 5.1**

$$\begin{aligned} & [X_n^-(0), X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1}} \\ &= q^{-n/2} [X_n^-(1), X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2} q} \\ &= (-1)^{n-1} [X_1^-(0), X_2^-(0), \dots, X_{n-1}^-(0), X_n^-(1)]_{q^{-1} q^{-1/2} \dots q^{-1/2}} \end{aligned}$$

*Proof.* From the Serre relations  $[X_i^-(0), X_j^-(1)]_{q^{(\alpha_i, \alpha_j)}} = -[X_j^-(0), X_i^-(1)]_{q^{(\alpha_i, \alpha_j)}}$ , it follows that

$$\begin{aligned} & [X_n^-(0), X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1}} \\ &= -[X_n^-(0), X_{n-1}^-(0), \dots, X_3^-(0), X_1^-(0), X_2^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1}} \\ &= (-1)^2 [X_n^-(0), X_{n-1}^-(0), \dots, X_1^-(0), X_2^-(0), X_3^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1}} \quad \text{inductively} \\ &= (-1)^{n-1} [X_1^-(0), X_2^-(0), \dots, X_{n-1}^-(0), X_n^-(1)]_{q^{-1} q^{-1/2} \dots q^{-1/2}} \\ &= q^{-1/2} (-1)^{n-2} [X_1^-(0), X_2^-(0), \dots, X_n^-(1), X_{n-1}^-(0)]_{q q^{-1/2} \dots q^{-1/2}} \\ &= q^{-n/2} [X_n^-(1), X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2} q} \end{aligned}$$

**Lemma 5.2** For  $i \leq n-1$  we have

$$\begin{aligned} & \left[ [X_1^-(0), X_2^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, \right. \\ & \left. [X_{i+1}^-(0), X_i^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{1/2}} = 0 \end{aligned}$$

*Proof.* Following [3], for  $y \in \text{End}(\tilde{\mathcal{F}})$  we define  $q$ -adjoint operators  $ad X_i^\pm(0) = S_i^\pm$  by

$$S_i^\pm(y) = X_i^\pm(0)y - K_i^{\pm 1} y K_i^{\mp 1} X_i^\pm(0).$$

We can directly check by Serre relations that

$$[S_i^\pm, S_j^\pm] = 0, \quad \text{if } A_{ij} = 0 \quad (28)$$

$$[S_i^\pm, S_i^\pm, S_j^\pm]_{q_i, q_i^{-1}} = 0, \quad \text{if } A_{ij} = -1 \quad (29)$$

$$S_i^{\pm 2}(X_j^\pm(m)) = 0 \quad \text{if } A_{ij} = -1 \quad (30)$$

We now prove the identity by induction on  $i$ .  $i=1$  is the Serre relation:

$$[X_1^-(0), X_2^-(0), X_1^-(1)]_{q^{-1/2} q^{1/2}} = -[X_1^-(0), X_1^-(0), X_2^-(1)]_{q^{-1/2} q^{1/2}} = 0.$$

Set

$$\begin{aligned} A' &= [X_1^-(0), X_2^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}} \\ A &= [X_{i-1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}} = S_{i-1}^- \dots S_2^-(X_1^-(1)). \end{aligned}$$

The inductive assumption is then

$$[A', [X_i^-(0), A]_{q^{-1/2}}]_{q^{1/2}} = 0.$$

Then we have

$$\begin{aligned} & [X_i^-(0), X_{i+1}^-(0), X_i^-(0), A]_{q^{-1/2} q^{-1/2} 1} = S_i^- S_{i+1}^- S_i^- S_{i-1}^- \dots S_2^-(X_2^-(1)) \\ &= \frac{1}{[2]_1} \left( S_i^{-2} S_{i+1}^- S_{i-1}^- \dots S_2^-(X_2^-(1)) + S_{i+1}^- (S_i^{-2} S_{i-1}^-) S_{i-2}^- \dots S_2^-(X_2^-(1)) \right) \\ &= \frac{1}{[2]_1} S_{i+1}^- \left( [2]_1 S_i^- S_{i-1}^- S_i^- S_{i-2}^- \dots S_2^-(X_2^-(1)) - S_{i-1}^- S_{i-1}^- S_i^{-2} S_{i-2}^- \dots S_2^-(X_1^-(1)) \right) \\ &= 0 \end{aligned} \tag{31}$$

where we have used (28-29). From (19) and  $[A', X_{i+1}^-(0)] = 0$  it follows that

$$\begin{aligned} & [A', X_i^-(0)]_{q^{-1/2}}, [X_{i+1}^-(0), X_i^-(0), A]_{q^{-1/2} q^{-1/2}} \Big]_{q^{1/2}} \\ &= [A', [X_i^-(0), X_{i+1}^-(0), X_i^-(0), A]_{q^{-1/2} q^{-1/2} 1}] + \\ & \quad [X_{i+1}^-(0), A', X_i^-(0), A]_{q^{-1/2} q^{1/2} q^{-1/2}}, X_i^-(0) \Big]_{q^{-1/2}} \\ &= 0 \quad \text{by (31) and induction hypothesis} \end{aligned}$$

□

**Lemma 5.3** *On the space  $End(\tilde{\mathcal{F}})$  we have*

$$[[\Phi_{\bar{1}}(z), X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, [X_{n-1}^-(0), \dots, X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{1/2}} = 0 \tag{32}$$

*Proof.* The proof is by induction and similar to that of Lemma 5.2, but with the  $q$ -adjoint operators  $T_i^\pm$ :  $T_i^\pm(y) = X_i^\pm(0)y - K_i^{\mp 1}yK_i^{\pm 1}X_i^\pm(0)$ . The operators  $T_i^\pm$  satisfy the same relations (28, 29, 30) as the operators  $S_i^\pm$ . Then we can express the commutators in the identities as follows:

$$\begin{aligned} [X_{n-1}^-(0), \dots, X_1^-(0), \Phi_{\bar{1}}(z)]_{q^{1/2} \dots q^{1/2}} &= T_{n-1}^- \dots T_1^-(\Phi_{\bar{1}}(z)), \\ [X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2}} &= T_{n-1}^- \dots T_2^-(X_1^-(0)). \end{aligned}$$

We claim that for  $i \leq n-1$

$$\begin{aligned} & T_i^- T_{i+1}^- T_i^- \dots T_1^-(\Phi_{\bar{1}}(z)) \\ &= [X_i^-(0), [X_{i+1}^-(0), \dots, X_1^-(0), \Phi_{\bar{1}}(z)]_{q^{1/2} \dots q^{1/2}}]_{q^{-\frac{1}{2}\delta_{n-1,i}}} \\ &= 0. \end{aligned} \tag{33}$$



The case of  $i = 1$  is exactly the Serre-like relation (cf. Prop. 4.3):

$$[X_1^-(0), X_1^-(0), \Phi_{\overline{1}}(z)]_{q^{1/2} q^{-1/2}} = 0.$$

Noting that  $T_i(\Phi_{\overline{1}}(z)) = 0$  for  $i \geq 2$ , we have

$$\begin{aligned} & T_i^- T_{i+1}^- T_i^- \cdots T_1^-(\Phi_{\overline{1}}(z)) \\ &= \frac{1}{[2]_1} (T_i^{-2} T_{i+1}^- + T_{i+1}^- T_i^{-2}) T_{i-1}^- \cdots T_1^-(\Phi_{\overline{1}}(z)) \\ &= \frac{1}{[2]_1} T_{i+1}^- ([2]_1 T_i^- T_{i-1}^- T_i^- - T_{i-1}^- T_i^{-2}) T_{i-2}^- \cdots T_1^-(\Phi_{\overline{1}}(z)) \\ &= 0. \end{aligned}$$

where we have used induction on  $i$ . Then we have for  $B = [X_{n-2}^-(0), \dots, X_1^-(0)]_{q^{1/2} \dots q^{1/2}}$

$$\begin{aligned} & \left[ [X_{n-1}^-(0), \dots, X_1^-(0)]_{q^{1/2} \dots q^{1/2}}, [X_{n-1}^-(0), \dots, X_1^-(0), \Phi_{\overline{1}}(z)]_{q^{1/2} \dots q^{1/2}} \right]_{q^{-1/2}} \\ &= \left[ X_{n-1}^-(0), \left[ B, [X_{n-1}^-(0), \dots, X_1^-(0), \Phi_{\overline{1}}(z)]_{q^{1/2} \dots q^{1/2}} \right] \right] + \\ & \quad \left[ X_{n-1}^-(0), [X_{n-1}^-(0), \dots, X_1^-(0), \Phi_{\overline{1}}(z)]_{q^{1/2} \dots q^{1/2}} \right]_{q^{-1/2}}, B \Big]_{q^{1/2}} \\ &= 0 \quad \text{by (19) and (33)} \end{aligned}$$

□

**Lemma 5.4** *For  $i \leq n - 2$  we have*

$$\begin{aligned} & \left[ [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, [X_{i+1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1/2}} \right] \\ &= -z q^{n+(1-i)/2} \left[ \Phi_{\overline{i+2}}(z), [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{-1}}. \end{aligned}$$

*Proof.* From Lemma 5.2 and (11) it follows that

$$\begin{aligned} & \left[ [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, [X_{i+1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1/2}} \right] \\ &= \left[ \left[ \Phi_{\overline{1}}(z), [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{-1/2}}, [X_{i+1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}} \right] \\ &= \left[ \Phi_{\overline{1}}(z), \left[ [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, [X_{i+1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{1/2}} \right]_{q^{-1}} \\ & \quad + q^{1/2} \left[ [\Phi_{\overline{1}}(z), X_{i+1}^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}}, [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{-1}} \\ &= q^{1/2} \left[ [X_{i+1}^-(0), \dots, X_2^-(0), \Phi_{\overline{1}}(z), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}}, [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{-1}} \\ &= -q^{n+1/2} z \left[ [X_{i+1}^-(0), \dots, X_2^-(0), \Phi_{\overline{2}}(z)]_{q^{-1/2} \dots q^{-1/2}}, [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}} \right]_{q^{-1}} \end{aligned}$$

where we used Lemma 5.2, the fact of  $\Phi_{\overline{1}}(z)$  commuting with  $X_i^-(0)$  for  $i \geq 2$  (cf. 11) and (27). Invoking the definition of  $\Phi_{\overline{i}}(z)$  and converting the  $q$ -commutators, we obtained the required identity. □

**Lemma 5.5** For  $i \leq n-2$  we have

$$\begin{aligned}
& [X_i^-(0), \dots, X_n^-(0), \dots, X_{i+1}^-(0), [\Phi_1^-(z), X_1^-(0), \dots, X_{i-1}^-(0)]_{q^{-1/2} \dots q^{-1/2}}, \\
& \quad [X_i^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}}]_{1 q^{-1/2} \dots q^{-1} \dots q^{-1/2} 1} \\
&= (-1)^{n-i-1} z q^{i/2+1} [X_i^-(0), \Phi_{i+1}(z), [X_1^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1} 1} \\
&= (-1)^{n-i} z q^{(i+1)/2} \left( [\Phi_i(z), [X_1^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} + \right. \\
& \quad \left. q^{1/2} [\Phi_{i+1}(z), [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \right).
\end{aligned}$$

*Proof.* It follows from Lemma 5.4 that

$$\begin{aligned}
& [X_i^-(0), \dots, X_n^-(0), \dots, X_{i+1}^-(0), [\Phi_1^-(z), X_1^-(0), \dots, X_{i-1}^-(0)]_{q^{-1/2} \dots q^{-1/2}}, \\
& \quad [X_i^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2}}]_{1 q^{-1/2} \dots q^{-1} \dots q^{-1/2} 1} \\
&= [X_i^-(0), \dots, X_n^-(0), \dots, X_{i+1}^-(0), -z q^{n+1-i/2} \cdot \\
& \quad [\Phi_{i+1}^-(z), [X_1^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1}}]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2} 1} \\
&= z q^{(n+1)/2} [X_i^-(0), \dots, X_{n-1}^-(0), \Phi_n(z), [X_1^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1} q^{-1/2} \dots q^{-1/2} 1} \\
&= (-1)^{n-i-1} z q^{1+i/2} [X_i^-(0), \Phi_{i+1}(z), [X_1^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1} 1}
\end{aligned}$$

where we have noted that  $X_{i+1}^-(0), \dots, X_n^-(0)$  commute with  $[X_1^-(0), \dots, X_{i-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}$  and the definition of  $\Phi_i(z)$ 's.  $\square$

Now we can prove our main relation.

*Proof of relation 9.* The idea is to move  $\Phi_1^-(z)$  to the right properly and use induction.

$$\begin{aligned}
& [\Phi_1^-(z), \hat{e}_0]_{q^{-1}} \\
&= \left[ [\Phi_1^-(z), X_1^-(0)]_{q^{-1/2}}, [X_2^-(0), \dots, X_n^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2}} \right]_{q^{-1/2}} - \\
& \quad q^{-1/2} [X_1^-(0), \dots, X_n^-(0), \dots, X_2^-(0), \Phi_2^-(z)]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2} q^{1/2}} z q^n \\
&= \left( [\Phi_1^-(z), X_1^-(0), X_2^-(0)]'_{q^{-1/2} q^{-1/2}}, \right. \\
& \quad [X_3^-(0), \dots, X_n^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2}}]_{q^{-1/2}} + \\
& \quad q^{-1/2} [X_2^-(0), \dots, X_n^-(0), \dots, X_3^-(0), \\
& \quad \left. [\Phi_1^-(z), X_1^-(0)]_{q^{-1/2}}, [X_2^-(0), X_1^-(1)]_{q^{-1/2}}] \right]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2} 1} + \\
& \quad (-1)^n z q^{1/2} [x_1^-(0), \Phi_2(z)]_{q^{1/2}} \\
&= [\Phi_1^-(z), X_1^-(0), X_2^-(0)]'_{q^{-1/2} q^{-1/2}}, \\
& \quad [X_3^-(0), \dots, X_n^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2}}]_{q^{-1/2}} +
\end{aligned}$$

$$(-1)^{n-2} z q^{3/2} [\Phi_3(z), [X_1^-(0), X_2^-(0)]_{q^{-1/2}}]_{q^{-1/2}}$$

where we used Lemma 5.5 and have cancelled the term  $z q^{1/2} [x_1^-(0), \Phi_2(z)]_{q^{1/2}}$ . Continuing this way for  $i$  steps we then have

$$\begin{aligned}
& [\Phi_{\overline{1}}(z), \hat{e}_0]_{q^{-1}} \\
&= \left[ [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, \right. \\
&\quad \left. [X_{i+1}^-(0), \dots, X_n^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1} \dots q^{-1/2}} \right]_{q^{-1/2}} \\
&\quad + (-1)^{n-i} z q^{(i+1)/2} [\Phi_{i+1}(z), [X_1^-(0), \dots, X_i^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \\
&= \left[ [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, [X_n^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1}} \right]_{q^{-1/2}} \\
&\quad + (-1)^{i+1} z q^{n/2} [\Phi_n(z), [X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \\
&= \left[ [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}, q^{n/2} [X_n^-(1), \dots, X_1^-(0)]_{q^{-1/2} \dots q^{-1/2} q^{-1}} \right]_{q^{-1/2}} + \\
&\quad - z q^{n/2} [\Phi_n(z), [X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \\
&= q^{-n/2} \left[ [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_{n-1}^-(0), X_n^-(1)]'_{q^{-1/2} \dots q^{-1/2} q^{-1}}, \right. \\
&\quad \left. [X_{n-1}^-(1), \dots, X_1^-(0)]_{q^{1/2} \dots q^{1/2}} \right]_{q^{3/2}} + \\
&\quad q^{-n/2-1} [X_n^-(1), [\Phi_{\overline{1}}(z), X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2} q^{-1}}, \\
&\quad [X_{n-1}^-(1), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{1/2} q^2} + \\
&\quad - z q^{n/2} [\Phi_n(z), [X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \quad \text{2nd term is 0 by lemma 5.3} \\
&= q^{-n/2} \left[ [\Phi_{\overline{1}}(z), [X_1^-(0), \dots, X_{n-1}^-(0), X_n^-(1)]'_{q^{-1/2} \dots q^{-1/2} q^{-1}} \right]_{q^{-1/2}}, \\
&\quad [X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{3/2}} + \\
&\quad - z q^{n/2} [\Phi_n(z), [X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \\
&= q^{-n/2} (-1)^{n-1} [\Phi_{\overline{1}}(z), X_n^-(0), \dots, X_2^-(0), X_1^-(1)]_{q^{-1/2} \dots q^{-1/2} q^{-1} q^{-1/2}}, \\
&\quad [X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{3/2}} + \\
&\quad - z q^{n/2} [\Phi_n(z), [X_1^-(0), \dots, X_{n-1}^-(0)]'_{q^{-1/2} \dots q^{-1/2}}]_{q^{-1/2}} \\
&= (-1)^n z [\Phi_n(z), [X_{n-1}^-(0), \dots, X_2^-(0), X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{3/2}} + \\
&\quad (-1)^n q z [\Phi_n(z), [X_{n-1}^-(0), \dots, X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{-1/2}} \\
&= (-1)^n z (1+q) [\Phi_n(z), [X_{n-1}^-(0), \dots, X_1^-(0)]_{q^{1/2} \dots q^{1/2}}]_{q^{1/2}} \\
&= (-1)^n z (1+q) \Phi_1(z)
\end{aligned}$$

where we have used  $[a, b]_{q^{3/2}} + q[a, b]_{q^{-1/2}} = (1+q)[a, b]_{q^{1/2}}$ .

□

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